

INDICES OF NORMALIZATION OF IDEALS

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ABSTRACT. We derive numerical estimates controlling the intertwined properties of the normalization of an ideal and of the computational complexity of general processes for its construction. In [8], this goal was carried out for equimultiple ideals via the examination of Hilbert functions. Here we add to this picture, in an important case, how certain Hilbert functions provide a description of the locations of the generators of the normalization of ideals of dimension zero. We also present a rare instance of normalization of a class of homogeneous ideals by a single colon operation.

1. INTRODUCTION

Let R be a Noetherian integral domain and let I be an ideal. The *normalization* of I is the integral closure in $R[t]$, \overline{A} , of the Rees algebra $A = R[It]$ of I . In case R is normal the nuance disappears. The properties of \overline{A} add significantly to an understanding of I and of the constructions it supports. The *index* terminology refers to the integers related to the description and construction of

$$\overline{A} = \sum_{n \geq 0} \overline{I}^n t^n = R[\overline{I}t, \dots, \overline{I}^{s_0} t^{s_0}].$$

In addition to the overall task of describing the generators and relations of \overline{A} , it includes the understanding of the following quantities:

- (i) Numerical indices for equalities of the type: find s such that

$$(\overline{A})_{n+s} = (A)_n \cdot (\overline{A})_s, \quad n \geq 0.$$

- (ii) Estimation of the number of steps that effective processes must traverse between A and \overline{A} ,

$$A = A_0 \subset A_1 \subset \dots \subset A_{r-1} \subset A_r = \overline{A}.$$

- (iii) Express r , s and s_0 in terms of invariants of A .
- (iv) Generators of \overline{A} : number of generators and distribution of their degrees in cases of interest.

These general questions acquire a high degree of specificity when $A = R[It]$, and the goal becomes the estimation of these indices in terms of invariants of I . A general treatment of item (ii) is given in [11] and [12]. For monomial ideals of finite colength a discussion is carried out in [19]. This paper is a sequel to [8], where some of the notions developed here originated. The focus in [8] was on deriving bounds on the coefficient $e_1(\overline{A})$ of the

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Hilbert function associated to ideals of finite co-length in local rings, and its utilization in the estimation of the length of general normalization algorithms. Here we introduce complementary notions and use them to address some of the same goals for more general ideals, but also show how known initial knowledge about the normalization allows us to give fairly detailed description of \overline{A} , particularly those affecting the distribution of its generators.

We now describe the organization of the paper. Section 2 gives the precise definitions of the indices outlined above (with one exception best left for Section 3), and describe some relationships amongst them. These indices acquire a sharp relief when the normalization $\sum_{n \geq 0} \overline{I^n} t^n$ is Cohen-Macaulay (Theorem 2.5). This result, whose proof follows *ipsis literis* the characterization of Cohen-Macaulayness in the Rees algebras of I -adic filtrations ([1], [6], [10]), has various consequences. It is partly used to motivate the treatment in Section 3 of the Sally module of the normalization algebra as a vehicle to study the number of generators and their degrees. In case the associated graded ring of the integral closure filtration \mathcal{F} of an \mathfrak{m} -primary ideal I , $\text{gr}_{\mathcal{F}}(R)$ has depth at least $\dim R - 1$, there are several positivity relations on the Hilbert coefficients, leading to descriptions of the distribution of the new generators (usually fewer as the degrees go up), and overall bounds for their numbers.

In Section 4, we present one of the rare instances where the normalization of the blowup ring is computed using an explicit expression as a colon ideal. Our formula applies to homogeneous ideals that are generated by forms of the same degree and satisfy some additional assumptions.

2. NORMALIZATION OF IDEALS

This section introduces auxiliary constructions and devices to examine the integral closure of ideals, and to study the properties and applications to normal ideals.

Indices of normalization. We begin by introducing some measures for the normalization of ideals. Suppose R is a commutative ring and J, I are ideals of R with $J \subset I$. J is a *reduction* of I if $I^{r+1} = JI^r$ for some integer r ; the least such integer is the *reduction number* of I relative to J . It is denoted $r_J(I)$. I is *equimultiple* if there is a reduction J generated by height I elements.

Definition 2.1. Let R be a locally analytically unramified normal domain and let I be an ideal.

- (i) The *normalization index* of I is the smallest integer $s = s(I)$ such that

$$\overline{I^{n+1}} = I \cdot \overline{I^n} \quad n \geq s.$$

- (ii) The *generation index* of I is the smallest integer $s_0 = s_0(I)$ such that

$$\sum_{n \geq 0} \overline{I^n} t^n = R[\overline{I}t, \dots, \overline{I^{s_0}}t^{s_0}].$$

- (iii) The *normal relation type* of I is the maximum degree of a minimal generating set of the presentation ideal

$$0 \rightarrow M \rightarrow R[T_1, \dots, T_m] \rightarrow R[\overline{I}t, \dots, \overline{I^{s_0}}t^{s_0}] \rightarrow 0.$$

For example, if $R = k[x_1, \dots, x_d]$ is a polynomial ring over a field and $I = (x_1^d, \dots, x_d^d)$, then $I_1 = \bar{I} = (x_1, \dots, x_d)^d$. It follows that $s_0(I) = 1$, while $s(I) = r_I(I_1) = d - 1$.

If (R, \mathfrak{m}) is a local ring, these indices have an expression in term of the special fiber ring F of the normalization map $A \rightarrow \bar{A}$.

Proposition 2.2. *With the above assumptions let*

$$F = \bar{A}/(\mathfrak{m}, It)\bar{A} = \sum_{n \geq 0} F_n.$$

We have

$$\begin{aligned} s(I) &= \sup\{n \mid F_n \neq 0\}, \\ s_0(I) &= \inf\{n \mid F = F_0[F_1, \dots, F_n]\}. \end{aligned}$$

Furthermore, if the index of nilpotency of F_i is r_i , then

$$s(I) \leq \sum_{i=1}^{s_0(I)} (r_i - 1).$$

Although these integers are well defined—since \bar{A} is finite over A —it is not clear, even in case R is a regular local ring, which invariants of R and of I have a bearing on the determination of $s(I)$. An affirmative case is that of a monomial ideal I of a ring of polynomials in d indeterminates over a field—when $s \leq d - 1$ (according to Corollary 2.6).

Equimultiple ideals. For primary ideals and some other equimultiple ideals there are relations between the two indices of normalization.

Proposition 2.3. *Let (R, \mathfrak{m}) be a local analytically unramified normal Cohen–Macaulay ring such that \mathfrak{m} is a normal ideal. Let I be \mathfrak{m} –primary ideal with multiplicity $e(I)$. Then*

$$s(I) \leq e(I)((s_0(I) + 1)^d - 1) - s_0(I)(2d - 1).$$

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Following Proposition 2.2, we estimate $s(I)$ (the Castelnuovo–Mumford regularity of F) in terms of the indices of nilpotency of the components F_n , for $n \leq s_0(I)$.

Let $J = (z_1, \dots, z_d)$ be a minimal reduction of I . For each component $I_n = \bar{I}^n$ of \bar{A} , we collect the following data:

$$\begin{aligned} J_n &= (z_1^n, \dots, z_d^n), \text{ a minimal reduction of } I_n \\ e(I_n) &= e(I)n^d, \text{ the multiplicity of } I_n \\ r_n = r_{J_n}(I_n) &\leq \frac{e(I_n)}{n}d - 2d + 1, \text{ a bound on the reduction number of } I_n. \end{aligned}$$

The last assertion follows from [17, Theorem 7.14], once it is observed that $I_n \subset \bar{\mathfrak{m}}^n = \mathfrak{m}^n$, by the normality of \mathfrak{m} .

We are now ready to estimate the index of nilpotency of the component F_n . With the notation above, we have $I_n^{r_n+1} = J_n I_n^{r_n}$. When this relation is read in F , it means that $r_n + 1 \geq \text{index of nilpotency of } F_n$.

Following Proposition 2.2, we have

$$s(I) \leq \sum_{n=1}^{s_0(I)} r_n = \sum_{n=1}^{s_0(I)} e(I)dn^{d-1} - s_0(I)(2d-1),$$

which we approximate with an elementary integral to get the assertion. \square

We can do considerably better when R is a ring of polynomials over a field of characteristic zero.

Theorem 2.4. *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field of characteristic zero and let I be a homogeneous ideal that is (x_1, \dots, x_d) -primary. One has*

$$s(I) \leq (e(I) - 1)s_0(I).$$

Proof. We begin by localizing R at the maximal homogeneous ideal and picking a minimal reduction J of I . We denote the associated graded ring of the filtration of integral closures $\{I_n = \overline{I}^n\}$ by G ,

$$G = \sum_{n \geq 0} I_n / I_{n+1}.$$

In this affine ring we can take for a Noether normalization a ring $A = k[z_1, \dots, z_d]$, where the z_i 's are the images in G_1 of a minimal set of generators of J .

There are two basic algebraic facts about the algebra G . First, its multiplicity as a graded A -module is the same as that of the associated graded ring of I , that is, $e(I)$. Second, since the Rees algebra of the integral closure filtration is a normal domain, so is the extended Rees algebra

$$C = \sum_{n \in \mathbb{Z}} I_n t^n,$$

where we set $I_n = R$ for $n \leq 0$. Consequently the algebra $G = C/(t^{-1})$ will satisfy the condition S_1 of Serre. This means that as a module over A , C is torsionfree.

We now apply the theory of Cayley-Hamilton equations to the elements of the components of G (see [15, Chapter 9]): For $u \in G_n$, we have an equation of integrality over A

$$u^r + a_1 u^{r-1} + \dots + a_r = 0,$$

where a_i are homogeneous forms of A , in particular $a_i \in A_{ni}$, and $r \leq e(G) = e(I)$. Since k has characteristic zero, using the argument of [15, Proposition 9.3.5], we obtain an equality

$$G_n^r = A_n G_n^{r-1}.$$

At the level of the filtration, this equality means that

$$I_n^r \subset J^n I_n^{r-1} + I_{nr+1},$$

which we weaken by

$$I_n^r \subset I \cdot I_{nr-1} + \mathfrak{m} I_{nr},$$

where we used $\overline{I^r} \subset (x_1, \dots, x_n) \overline{I^{r-1}}$. Finally, in F , this equation shows that the indices of nilpotency of the components F_n are bounded by $e(I)$, as desired. Now we apply Proposition 2.2 (and delocalize back to the original homogeneous ideals). \square

Cohen-Macaulay normalization. Expectably, normalization indices are easier to obtain when the normalization of the ideal is Cohen-Macaulay. The following is directly derived from the known characterizations of Cohen-Macaulayness of Rees algebras of ideals in terms of associated graded rings and reduction numbers ([1], [6], [10]).

Theorem 2.5. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring and let $\{I_n, n \geq 0\}$ be a decreasing multiplicative filtration of ideals, with $I_0 = R$, $I_1 = I$, and the property that the corresponding Rees algebra $B = \sum_{n \geq 0} I_n t^n$ is finite over A . Suppose that $\text{height } I \geq 1$ and let J be a minimal reduction of I . If B is Cohen-Macaulay, then*

$$I_{n+1} = JI_n = I_1 I_n \quad \text{for every } n \geq \ell(I_1) - 1,$$

and in particular, B is generated over $R[It]$ by forms of degrees at most $\ell - 1 = \ell(I_1) - 1$,

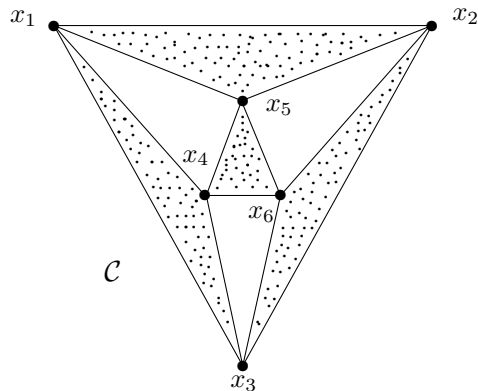
$$\sum_{n \geq 0} I_n t^n = R[I_1 t, \dots, I_{\ell-1} t^{\ell-1}].$$

The proof of Theorem 2.5 relies on substituting in any of the proofs mentioned above ([1, Theorem 5.1], [6, Theorem 2.3], [10, Theorem 3.5]) the I -adic filtration $\{I^n\}$ by the filtration $\{I_n\}$.

Corollary 2.6. *Let (R, \mathfrak{m}) be a local analytically unramified normal Cohen-Macaulay ring and let I be an ideal. If \overline{A} is Cohen-Macaulay then both indices of normalization $s(I)$ and $s_0(I)$ are at most $\ell(I) - 1$. Moreover, if I^n is integrally closed for $n < \ell(I)$, then I is normal.*

A case this applies to is that of monomial ideals in a polynomial ring, since the ring \overline{A} is Cohen-Macaulay by Hochster's theorem ([2, Theorem 6.3.5]) (see also [9]).

Example 2.7. Let $I = I(\mathcal{C}) = (x_1 x_2 x_5, x_1 x_3 x_4, x_2 x_3 x_6, x_4 x_5 x_6)$ be the edge ideal associated to the clutter



Consider the incidence matrix A of this clutter, i.e., the columns of A are the exponent vectors of the monomials that generate I . Since the polyhedron $Q(A) = \{x | xA \geq 1; x \geq 0\}$ is integral, we have the equality $\overline{R[It]} = R_s(I)$, the symbolic Rees algebra of I (see [3, Proposition 3.13]). The ideal I is not normal because the monomial $x_1 x_2 \cdots x_6$ is in $\overline{I^2} \setminus I^2$.

The minimal primes of I are:

$$\begin{aligned} \mathfrak{p}_1 &= (x_1, x_6), & \mathfrak{p}_2 &= (x_2, x_4), & \mathfrak{p}_3 &= (x_3, x_5), \\ \mathfrak{p}_4 &= (x_1, x_2, x_5), & \mathfrak{p}_5 &= (x_1, x_3, x_4), & \mathfrak{p}_6 &= (x_2, x_3, x_6), & \mathfrak{p}_7 &= (x_4, x_5, x_6). \end{aligned}$$

For any n ,

$$I^{(n)} = \bigcap_{i=1}^7 \mathfrak{p}_i^n.$$

A computation with *Macaulay 2* ([4]) gives that $\overline{I^2} = (I^2, x_1x_2 \cdots x_6)$ and that $\overline{I^3} = I \overline{I^2}$. By Theorem 2.5, $I^{(n)} = II^{(n-1)}$ for $n \geq \ell(I) = 4$, where $\ell(I)$ is the analytic spread of I . As a consequence,

$$\overline{R[It]} = R[It, x_1x_2x_3x_4x_5x_6t^2].$$

Question 2.8. Given the usefulness of Theorem 2.5, it would be worthwhile to look at the situation short of Cohen–Macaulayness. For the integral closure of a standard graded algebra A of dimension d , it was possible in [14] to derive degree bounds assuming only S_{d-1} for \overline{A} . Another issue is to compute the relation type of A in Theorem 2.5.

3. SALLY MODULES AND NORMALIZATION OF IDEALS

Let (R, \mathfrak{m}) be an analytically unramified local ring of dimension d and I an \mathfrak{m} -primary ideal. Let $\mathcal{F} = \{I_n, n \geq 0\}$ be a decreasing, multiplicative filtration of ideals, with $I_0 = R$, $I_1 = I$, with the property that the corresponding Rees algebra

$$\mathcal{R} = R(\mathcal{F}) = \sum_{n \geq 0} I_n t^n$$

is a Noetherian ring. We will examine in detail the case when \mathcal{F} is a subfiltration of the integral closure filtration of the powers of I , $I_n \subset \overline{I^n}$.

There are several algebraic structures attached to \mathcal{F} , among which we single out the associated graded ring of \mathcal{F} and its Sally modules. The first is

$$\mathrm{gr}_{\mathcal{F}}(R) = \sum_{n \geq 0} I_n / I_{n+1},$$

whose properties are closely linked to $R(\mathcal{F})$. We note that if J is a minimal reduction of I_1 , then $\mathrm{gr}_{\mathcal{F}}(R)$ is a finite generated module over $\mathrm{gr}_J(R)$, so that it is a semi-standard graded algebra.

To define the Sally module, we choose a minimal reduction J of I (if need be, we may assume that the residue field of R is infinite). Note that \mathcal{R} is a finite extension of the Rees algebra $\mathcal{R}_0 = R[Jt]$ of the ideal J . The corresponding Sally module S defined by the exact sequence of finitely generated modules over \mathcal{R}_0 ,

$$(1) \quad 0 \rightarrow I\mathcal{R}_0 \longrightarrow \mathcal{R}_+ [+1] \longrightarrow S = \bigoplus_{n=1}^{\infty} I_{n+1} / J^n I \rightarrow 0.$$

As an \mathcal{R}_0 -module, S is annihilated by an \mathfrak{m} -primary ideal. If $S \neq 0$, $\dim S \leq d$, with equality if R is Cohen-Macaulay. The Artinian module

$$S/JtS = \bigoplus_{k \geq 1} I_{k+1} / JI_k$$

gives some control over the number of generators of \mathcal{R} as an \mathcal{R}_0 -module. If S is Cohen-Macaulay, this number is also its multiplicity. It would, of course, be more useful to obtain bounds for the length of $\mathcal{R}/(\mathfrak{m}, \mathcal{R}_+)\mathcal{R}$, but this requires lots more.

The cohomological properties of \mathcal{R} , $\text{gr}_{\mathcal{F}}(R)$ and S become more entwined when R is Cohen-Macaulay. Indeed, under this condition, the exact sequence (1) and the exact sequences (originally paired in [5]):

$$(2) \quad 0 \rightarrow \mathcal{R}_+[+1] \longrightarrow \mathcal{R} \longrightarrow \text{gr}_{\mathcal{F}}(R) \rightarrow 0$$

$$(3) \quad 0 \rightarrow \mathcal{R}_+ \longrightarrow \mathcal{R} \longrightarrow R \rightarrow 0,$$

with the tautological isomorphism

$$\mathcal{R}_+[+1] \cong \mathcal{R}_+$$

gives a fluid mechanism to pass cohomological information around.

Proposition 3.1. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d and \mathcal{F} a filtration as above. Then*

- (a) $\text{depth } \mathcal{R} \leq \text{depth } \text{gr}_{\mathcal{F}}(R) + 1$, with equality if $\text{gr}_{\mathcal{F}}(R)$ is not Cohen-Macaulay.
- (b) $\text{depth } S \leq \text{depth } \text{gr}_{\mathcal{F}}(R) + 1$, with equality if $\text{gr}_{\mathcal{F}}(R)$ is not Cohen-Macaulay.

Proof. For (a), see [17, Theorem 3.25]. For (b), it follows simply because $I\mathcal{R}_0$ is a maximal Cohen-Macaulay \mathcal{R}_0 -module. \square

Hilbert functions. Another connection between \mathcal{F} and S is realized via their Hilbert functions. Set

$$H_{\mathcal{F}}(n) = \lambda(R/I_n), \quad H_S(n-1) = \lambda(I_n/IJ^{n-1}).$$

The associated Poincaré series

$$P_{\mathcal{F}}(t) = \frac{f(t)}{(1-t)^{d+1}},$$

$$P_S(t) = \frac{g(t)}{(1-t)^d}$$

are related by

$$\begin{aligned} P_{\mathcal{F}}(t) &= \frac{\lambda(R/J) \cdot t}{(1-t)^{d+1}} + \frac{\lambda(R/I)(1-t)}{(1-t)^{d+1}} - P_S(t) \\ &= \frac{\lambda(R/I) + \lambda(I/J) \cdot t}{(1-t)^{d+1}} - P_S(t). \end{aligned}$$

The proof of this fact follows as in [16] and [18], replacing the I -adic filtration by the filtration \mathcal{F} .

Proposition 3.2. *The h -polynomials $f(t)$ and $g(t)$ are related by*

$$(4) \quad f(t) = \lambda(R/I) + \lambda(I/J) \cdot t - (1-t)g(t).$$

In particular, if $f(t) = \sum_{i \geq 0} a_i t^i$ and $g(t) = \sum_{i \geq 1} b_i t^i$, then for $i \geq 2$

$$a_i = b_{i-1} - b_i.$$

Corollary 3.3. *If $\text{gr}_{\mathcal{F}}(R)$ is Cohen-Macaulay, then the h -vector of S is positive and non-increasing,*

$$b_i \geq 0, \quad b_1 \geq b_2 \geq \cdots \geq 0.$$

In particular, if $b_{k+1} = 0$ for some k , then \mathcal{R} is generated by its elements of degree at most k .

Proof. That $b_i \geq 0$ follows because S is Cohen-Macaulay, while the positivity of the a_i 's for the same reason and the difference relation shows that $b_{i-1} \geq b_i$. For the other assertion, since S is Cohen-Macaulay, $b_k = \lambda(I_{k+1}/JI_k)$. The proof of this fact is a modification of [18, See 1.1], using the filtration \mathcal{F} instead of the I -adic filtration. Therefore if b_k vanishes no fresh generators for I_{k+1} are needed. \square

Remark 3.4. The equality (4) has several useful general properties, of which we remark the following. For $k \geq 2$, one has

$$f^{(k)}(1) = kg^{(k-1)}(1),$$

that is the coefficients e_i of the Hilbert polynomials of $\text{gr}_F(R)$ and S are identical, more precisely

$$e_{i+1}(\mathcal{F}) = e_i(S), \quad i \geq 1.$$

Observe that when $\text{depth } \text{gr}_{\mathcal{F}}(R) \geq d-1$, S is Cohen-Macaulay so its h -vector is positive, and therefore all the e_i along with it (see [7, Corollary 2]).

Corollary 3.5. *If $\text{gr}_{\mathcal{F}}(R)$ is Cohen-Macaulay and $g(t)$ is a polynomial of degree at most 4, then*

$$e_2(\mathcal{F}) \geq e_3(\mathcal{F}) \geq e_4(\mathcal{F}) \geq e_5(\mathcal{F}).$$

Proof. By our assumption

$$g(t) = b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4.$$

As

$$e_{k+1}(\mathcal{F}) = e_k(S) = \frac{g^{(k)}(1)}{k!},$$

we have the following equations:

$$\begin{aligned} e_2(\mathcal{F}) &= b_1 + 2b_2 + 3b_3 + 4b_4 \\ e_3(\mathcal{F}) &= b_2 + 3b_3 + 6b_4 \\ e_4(\mathcal{F}) &= b_3 + 4b_4 \\ e_5(\mathcal{F}) &= b_4 \end{aligned}$$

Now the assertion follows because $b_1 \geq b_2 \geq b_3 \geq b_4 \geq 0$, according to Corollary 3.3. \square

This considerably lowers the possible number of distinct Hilbert functions for such algebras.

Remark 3.6. The assumptions of Corollary 3.5 are satisfied for instance if $\dim R \leq 6$ and \mathcal{R} is Cohen-Macaulay.

These relations provide a fruitful ground for several questions. Let (R, \mathfrak{m}) be a local Nagata domain, I an \mathfrak{m} -primary ideal and suppose \mathcal{F} is the integral closure filtration of I . How generally does the inequality

$$\overline{e}_1(I) \geq e_0(I) - \lambda(R/\overline{I})$$

hold? Of course it is true if $\text{gr}_{\mathcal{F}}(R)$ is Cohen-Macaulay, and possibly if the Sally module S is Cohen-Macaulay.

Number of generators. Another application of Sally modules is to obtain a bound for the number of generators (and the distribution of their degrees) of \mathcal{R} as an \mathcal{R}_0 -algebra. (Sometimes the notation is used to denote the number of module generators.) A distinguished feature is the front loading of the new generators in the Cohen-Macaulay case.

Theorem 3.7. *Let \mathcal{F} be a filtration as above.*

- (a) *If $\text{depth } \text{gr}_{\mathcal{F}}(R) \geq d - 1$, the \mathcal{R}_0 -algebra \mathcal{R} can be generated by $e_1(\mathcal{F})$ elements.*
- (b) *If $\text{depth } \text{gr}_{\mathcal{F}}(R) = d$, the \mathcal{R}_0 -algebra \mathcal{R} can be generated by $e_0(\mathcal{F})$ elements.*
- (c) *If \mathcal{R} is Cohen-Macaulay, it can be generated by $\lambda(\mathcal{F}_1/J) + (d - 2)\lambda(\mathcal{F}_2/\mathcal{F}_1J)$ elements.*

Proof. (a) From the relation (4), we have

$$e_1(\mathcal{F}) = f'(1) = \lambda(F_1/J) + g(1).$$

From the sequence (1) that defines S , one has

$$\nu(\mathcal{R}) \leq \nu(F_1/J) + \nu(S) \leq \lambda(F_1/J) + \nu(S).$$

Since S is Cohen-Macaulay, $\nu(S) = e_0(S) = g(1)$, which combine to give the promised assertion.

(b) A generating set for \mathcal{R} can be obtained from a lift of a minimal set of generators for $\text{gr}_{\mathcal{F}}(R)$, which is given by its multiplicity since it is Cohen-Macaulay.

(c) Since \mathcal{R} is Cohen-Macaulay, its reduction number is $\leq d - 1$. Thus the h -polynomial of $\text{gr}_{\mathcal{F}}(R)$ has degree $\leq d - 1$, and consequently the h -polynomial of the Sally module has degree at most $d - 2$. As the h -vector of S is decreasing, its multiplicity is at most $(d - 2)\lambda(\mathcal{F}_2/\mathcal{F}_1J)$, and we conclude as in (a). \square

Remark 3.8. A typical application is to the case $d = 2$ with \mathcal{F} being the filtration $F_n = \overline{I}^n$.

Remark 3.9. There are several relevant issues here. The first, to get bounds for $e_1(\mathcal{F})$. This is addressed in [8]. For instance, when R is a regular local ring of characteristic zero,

$$e_1(\mathcal{F}) \leq \frac{(d-1)e_0}{2}.$$

4. ONE-STEP NORMALIZATION OF REES ALGEBRAS

In this section we present one of the rare instances where the normalization of the Rees ring can be computed in a single step using an explicit expression as a colon ideal. Our formula applies to homogeneous ideals that are generated by forms of the same degree and satisfy some additional assumptions.

The G_d assumption in the theorem means that the minimal number of generators $\nu(I_{\mathfrak{p}})$ is at most $\dim R_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} containing I with $\dim R_{\mathfrak{p}} \leq d-1$. In the proof of the theorem we use the theory of residual intersections. Let s be an integer with $s \geq \text{height } I$. Recall that $\mathfrak{a} : I$ is a s -residual intersection of I if \mathfrak{a} is an s -generated R -ideal properly contained in I and $\text{ht } \mathfrak{a} : I \geq s$.

Theorem 4.1. *Let k be an infinite field, $R = k[x_1, \dots, x_d]$ a positively graded polynomial ring and I an ideal of height g generated by forms of degree δ , and set $A = R[It]$. Assume I satisfies G_d , $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq d - g$, and I is normal locally on the punctured spectrum. Let J be a homogeneous minimal reduction of I and write $\sigma = \sum_{i=1}^d \deg x_i$. Then*

$$\overline{A} = R[Jt] :_{R[t]} R_{\geq g\delta - \delta - \sigma + 1};$$

in particular $\overline{I} = J :_R R_{\geq g\delta - \delta - \sigma + 1}$. Furthermore, I is a normal ideal of linear type if and only if $\delta \leq \frac{\sigma-1}{g-1}$ or $\nu(I) \leq d-1$

Proof. We may assume $g \geq 1$ and $d \geq 2$. Notice that J is of linear type and $R[Jt]$ is Cohen-Macaulay [13]. Write \mathfrak{m} for the homogeneous maximal ideal of R .

If $\ell(I) < d$ then $I = J$. Since $A = R[Jt]$ is Cohen-Macaulay, I is normal on the punctured spectrum, and $\text{ht } \mathfrak{m}A \geq 2$, it follows that $\overline{A} = A = R[Jt]$. On the other hand, $R[Jt] :_{R[t]} R_{\geq g\delta - \delta - \sigma + 1} = R[Jt]$ because $\text{ht}(R_{\geq g\delta - \delta - \sigma + 1})R[Jt] \geq \text{ht } \mathfrak{m}R[Jt] \geq 2$ and $R[Jt]$ is Cohen-Macaulay.

Thus we may assume $\ell(I) = d$. Write $\mathfrak{b} = R_{\geq g\delta - \delta - \sigma + 1}$. Notice that $R[Jt] :_{R[t]} \mathfrak{b} = R[Jt] :_{\text{Quot}(R[t])} \mathfrak{b}$ since $d \geq 2$. As the two $R[Jt]$ -modules \overline{A} and $R[Jt] : \mathfrak{b}$ satisfy S_2 and as locally on the punctured spectrum of R , I is of linear type and normal, it suffices to prove the equality $\overline{A}_{\mathfrak{m}R[Jt]} = (R[Jt] : \mathfrak{b})_{\mathfrak{m}R[Jt]}$.

Let f_1, \dots, f_d be a general generating set of J consisting of forms of degree δ and let φ be a minimal presentation matrix of f_1, \dots, f_d . Notice that the entries along any column of φ are forms of the same degree. One has $R[Jt] \cong R[T_1, \dots, T_d]/I_1(\underline{T}\varphi)$. Let $K = k(T_1, \dots, T_d)$ and $B = K[x_1, \dots, x_d]/I_1(\underline{T}\varphi)$. Notice that $B_{\mathfrak{m}B} \cong R[Jt]_{\mathfrak{m}R[Jt]}$. Since B is a positively graded K -algebra with irrelevant maximal ideal, we conclude that B is a domain of dimension one. In the ring $K[x_1, \dots, x_d]$, $I_1(\underline{T}\varphi) = \mathfrak{a} : J$ is a $(d-1)$ -residual intersection of J , hence of I , where \mathfrak{a} is generated by $d-1$ forms of degree δ . From this we conclude that $\omega_B \cong I^{d-g}/\mathfrak{a}I^{d-g-1}((d-1)\delta - \sigma)$ (see [13]). Thus $a(B) = (g-1)\delta - \sigma$.

Since B is a positively graded K -domain, it follows that \overline{B} is a positively graded L -domain for some finite field extension L of K . As $\dim B = 1$, \overline{B} is a principal ideal domain, hence $\overline{B} = L[t]$ for some homogeneous element t of degree $\alpha > 0$. Since the Hilbert function of \overline{B} as a B -module is constant in degrees divisible by α and zero otherwise, the conductor of B is of the form $B : \overline{B} = B_{\geq \varepsilon}$ for some ε , where $\varepsilon = \max \{i \mid [\overline{B}/B]_i \neq 0\} + 1$. The sequence

$$0 \longrightarrow B \longrightarrow \overline{B} \longrightarrow \overline{B}/B \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \overline{B}/B \longrightarrow H_{\mathfrak{m}B}^1(B) \longrightarrow H_{\mathfrak{m}B}^1(\overline{B}) \longrightarrow 0.$$

If $\overline{B}/B \neq 0$, then $a(B) \geq 0$ since \overline{B}/B is concentrated in non-negative degrees. On the other hand $a(\overline{B}) = -\alpha < 0$. Thus $\varepsilon = a(B) + 1 = g\delta - \delta - \sigma + 1$. Hence $B : \overline{B} = B_{\geq g\delta - \delta - \sigma + 1}$. If on the other hand $\overline{B}/B = 0$, then $a(B) = a(\overline{B})$, hence $g\delta - \delta - \sigma = -\alpha < 0$. Thus $B_{\geq g\delta - \delta - \sigma + 1} = B = B : \overline{B}$. Therefore in either case $B : \overline{B} = B_{\geq g\delta - \delta - \sigma + 1} = \mathfrak{b}B$, or equivalently, $\overline{B} = B : \mathfrak{b}B$. Localizing at $\mathfrak{m}B$ we conclude that $\overline{R[It]}_{\mathfrak{m}R[It]} = \overline{R[Jt]}_{\mathfrak{m}R[Jt]} = (R[Jt] : \mathfrak{b})_{\mathfrak{m}R[Jt]}$. \square

Remark 4.2. Notice in the previous theorem if R is standard graded then $R_{\geq g\delta - \delta - \sigma + 1} = \mathfrak{m}^{g\delta - \delta - \sigma + 1}$

Remark 4.3. In the presence of the G_d assumption the depth conditions in Theorem 4.1 are satisfied, for example if I is perfect of height 2, I is Gorenstein of height 3, or more generally I is licci.

Another class of ideals satisfying the assumptions of the theorem are 1-dimensional ideals:

Example 4.4. Let k be an infinite field, $R = k[x_1, \dots, x_d]$ a standard graded polynomial ring with homogeneous maximal ideal \mathfrak{m} , I a 1-dimensional reduced ideal generated by forms of degree δ , and J an ideal generated by d general forms of degree δ in I . Then for every n ,

$$\overline{I^n} = J^n : \mathfrak{m}^{(d-2)(\delta-1)-1}.$$

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